2. A standard die is rolled repeatedly. Let

- X_n be the largest value seen in the first n rolls,
- S_n be the number of 6s seen in the first n rolls,
- T_n be the number of 6s seen on rolls $n-1$ and n.

(so $T_n = S_n - S_{n-2}$).
Which of $X_n,\ S_n,\ T_n$ are Markov chains?
For those that are draw their transition graphs.

2. We start with X_n. Let's work out the transition probabilities for all pairs of states. It is clear that

$$\mathbb{P}(X_{n+1} = j | X_n = i) = 0 \qquad \text{if } i > j.$$

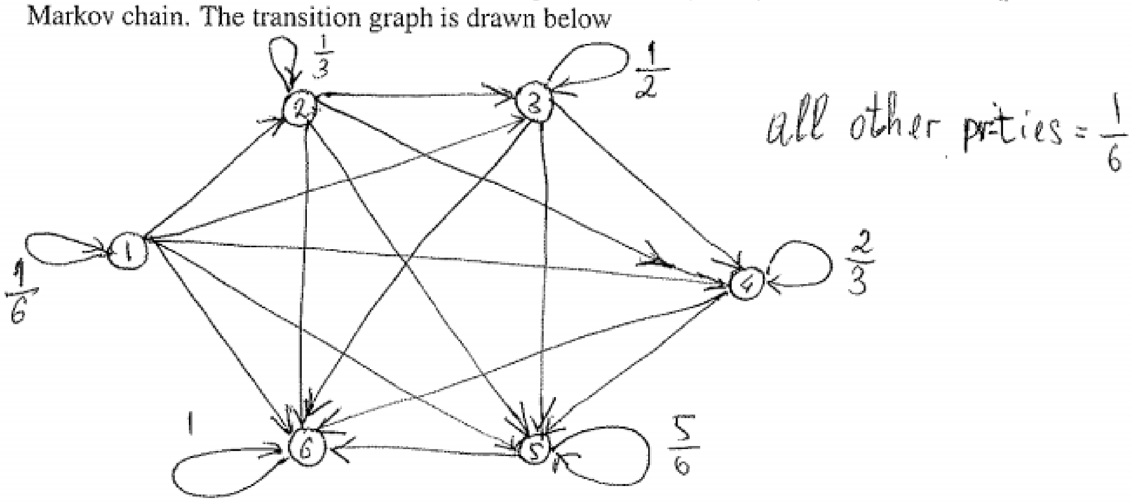
If $X_n = i$ then $X_{n+1} = i$ if and only if throw number $(n+1)$ is no more than i. Thus,

$$\mathbb{P}(X_{n+1} = i | X_n = i) = \frac{i}{6}.$$

If $i < j$, and $X_n = i$ then $X_{n+1} = j$ if and only if throw number $(n+1)$ is a j. Thus,

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \frac{1}{6} \qquad \text{if } i < j.$$

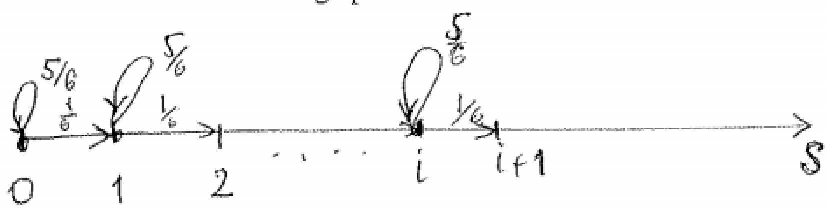
Since these transition probabilities do not depend on $X_{n-1}, X_{n-2}, \ldots$ we deduce that X_n is a Markov chain. The transition graph is drawn below



Next, we consider S_n. If throw number $(n+1)$ is not a 6 then he number of 6s seen on the first $(n+1)$ rolls is the number of sixes seen in the first n rolls. If throw number $(n+1)$ is a 6 then it increases by 1. Thus

$$\mathbb{P}(S_{n+1} = j | S_n = i) = \begin{cases} \frac{5}{6} & \text{if } i = j \\ \frac{1}{6} & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Again, these transition probabilities do not depend on $S_{n-1}, S_{n-2}, \ldots$. We deduce that S_n is a Markov chain. The transition graph is drawn below



Finally, T_n is not a Markov chain. To show this we must show that the Markov condition is violated. Suppose that $T_1 = 1, T_2 = 1$; in this case the first throw must have been a 6 and the second throw must have been something else. Thus T_3 is at most 1. It follows that

$$\mathbb{P}(T_3 = 2 | T_2 = 1, T_1 = 1) = 0.$$

Suppose that $T_1 = 0, T_2 = 1$; in this case the first throw must have been something other than a 6 and the second throw must have been a 6. Thus, if the third throw is a 6 then $T_3 = 2$. It follows that,

$$\mathbb{P}(T_3 = 2 | T_2 = 1, T_1 = 0) = \frac{1}{6}.$$

We deduce that $\mathbb{P}(T_3 = 2 | T_2 = 1)$ does depend on T_1 and hence the Markov property does not hold.

4. Initially, an urn contains 8 balls of which 4 are red and 4 are green. Two balls are selected from the urn. If one of them is red and the other is green then they are discarded and replaced by two blue balls. In all other cases the selected balls are returned to the urn. This process repeats until the urn contains only blue balls. Let X_n denote the number of red balls in the urn after n draws. Show that X_n is a Markov chain and find its transition matrix.

4. Observe that rules of the game are such that the number of green balls is always equal to the number of red balls. If $X_n = i$ then either $X_{n+1} = i$ or $X_{n+1} = i-1$. We now have for $i = 0, 1, 2, 3, 4$, that

$$P\{X_{n+1} = i-1 \,|\, X_n = i\} = P\{\text{ choose one of } i \text{ red and one of } i \text{ green balls }\} = \frac{i^2}{\binom{8}{2}} = \frac{i^2}{28}$$

and hence

$$P\{X_{n+1} = i \,|\, X_n = i\} = 1 - \frac{i^2}{28}.$$

The transition matrix $(P_{ij})_{0 \le i,j \le 4}$ now looks as follows

$$\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{28} & \frac{27}{28} & 0 & 0 & 0 \\ 0 & \frac{1}{7} & \frac{6}{7} & 0 & 0 \\ 0 & 0 & \frac{9}{28} & \frac{19}{28} & 0 \\ 0 & 0 & 0 & \frac{4}{7} & \frac{3}{7} \end{pmatrix}$$

4. Five balls are distributed between two urns, labelled A and B. Each period, an urn is selected at random, and if it is not empty, a ball from that urn is removed and placed into the other urn. In the long run, what fraction of time, on the average, is urn A empty.

4. Let X_n be the number of balls in urn A. It is clear that the distribution of X_{n+1} depends only on the value of X_n (and does not depend on the values of $X_k,\ k \le n-1$. Hence X_n is a MC. Since each urn can be chosen with probability 0.5, the transition probabilities are given by

$$p_{i,i+1} = p_{i,i-1} = \frac{1}{2} \text{ if } i = 1, 2, 3, 4; \quad p_{0,1} = p_{0,0} = p_{5,4} = p_{5,5} = \frac{1}{2}$$

or, equivalently, by

$$\mathbb{P} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

It is easy to check that the equilibrium distribution is given by $\underline{w} = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. Hence, for large values of time n, the fraction of time the MC spends in the state 0 (urn A is empty) is $\frac{n_0}{n} \simeq \frac{1}{6}$.

3. A Markov chain on states 0,1,2,3,4, and 5 has the following transition probability matrix:

$$\begin{Vmatrix} 1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 3/4 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 4/5 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{Vmatrix}.$$

Find all communicating (equivalence) classes.

3. It is evident from the transition matrix that $0 \leftrightarrow 2$, $4 \leftrightarrow 5$, and $1 \leftrightarrow 3$. Since $p_{00} + p_{04} = 1$ and $p_{01} + p_{44} = 1$, it is impossible to reach from any of the states $\{0, 2\}$ any of the other states. Similarly, it is impossible to reach from any of the states $\{4, 5\}$ any of the other states.

Thus, the state space is partitioned into three equivalence classes: $\{0, 2\}$, $\{4, 5\}$, and $\{1, 3\}$.

Note that $\{0, 2\}$, $\{1, 3\}$ are two classes of recurrent states whereas $\{4, 5\}$ is a transient class. Why?

1. Suppose that customers arrive at a facility according to a Poisson process of rate $\lambda = 2$ customers per hour. Let $X(t)$ be the number of customers that have arrived up to time t. Determine the following probabilities and conditional probabilities:

   (a) $P(X(1) = 2)$
   (b) $P(X(1) = 2, X(3) = 6)$
   (c) $P(X(1) = 2 | X(3) = 6)$
   (d) $P(X(3) = 6 | X(1) = 2)$

   Give details of your calculations.

2. For a Poisson process $X(t)$ of rate λ and two fixed times t_1 and t_2, $t_1 < t_2$, find the joint probability mass function for the two random variables $X(t_1)$ and $X(t_2)$.

1. (a)
$$P(X(1)=2) = P(X(1)-X(0)=2) = e^{-\lambda t}\frac{(\lambda t)^2}{2!}\bigg|_{\lambda=2,t=1} = 2e^{-2} = 0.2707$$
by (iii) and (ii) in the definition of the Poisson process.

(b) By the definition of the Poisson process, its increments are independent random variables. Hence
$$P(X(1)=2, X(3)=6) = P(X(1)=2, X(3)-X(1)=4) = P(X(1)=2)\,P(X(3)-X(1)=4)$$
and therefore
$$P(X(1)=2, X(3)=6) = e^{-\lambda}\frac{\lambda^2}{2!}e^{-\lambda(3-1)}\frac{(\lambda(3-1))^4}{4!} = \frac{64}{3}e^{-6} = 0.05288,$$

(c) By the theorem proved in lectures, $X(1)$ conditioned on $X(3)=6$ is a Binomial random variable with parameters $(6, \frac{1}{3})$. Hence
$$P(X(1)=2|X(3)=6) = \binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4 = \frac{80}{243} = 0.3292$$

(d) Since $P(X(1)=2, X(3)=6) = P(X(1)=2)\times P(X(3)-X(1)=4)$, we have
$$P(X(3)=6|X(1)=2) = \frac{P(X(1)=2, X(3)=6)}{P(X(1)=2)} = P(X(3)-X(1)=4) = \frac{32e^{-4}}{3} = 0.1953$$

2. We must find $P(X(t_1)=k, X(t_2)=n)$ for all values $k, n = 0, 1, 2\ldots.$
$$P(X(t_1)=k, X(t_2)=n) = 0 \quad \text{for all } k > n.$$
This is because $t_1 < t_2$ ($X(t)$ counts events that happened in the interval $(0, t]$, hence $X(t)$ cannot decrease with t).

When $n \geq k$
$$P\{X(t_1)=k,\ X(t_2)=n\} = P\{X(t_1)=k\}\times P\{X(t_2)-X(t_1)=n-k\} = \frac{t_1^k(t_2-t_1)^{n-k}\lambda^n e^{-\lambda t_2}}{k!(n-k)!}.$$

For the interactive system in Figure 6.14, suppose that we are given the following information:
mean user think time = 5 seconds
expected service time at device i = .01 seconds
utilization of device i = .3
utilization of CPU = .5
expected number of visits to device i per visit to CPU = 10
expected number of jobs in the central subsystem (the cloud shape) = 20
expected total time in system (including think time) per job = 50 seconds.
How many jobs are there in the queue portion of the CPU on average, $\mathbf{E}\, N_{Q\text{cpu}}$?

For the following system, suppose that we are given the following information:

mean user think time = 5 seconds

expected service time at device i = 0.01 seconds

utilization of device i per visit to CPU = 10

expected number of jobs in the central subsystem (the cloud shape) = 20
expected total time in system (including think time) per job = 50 seconds.

How many jobs are there in the queue portion of the CPU on average

Final answer:

The **average number** of **jobs** in the queue portion of the CPU is approximately 10.002 jobs.

Explanation:

To calculate the **average number** of **jobs** in the *queue portion* of the *CPU*, we can use *Little's Law*. Little's Law states that the average number of jobs in a system is equal to the *average arrival rate* multiplied by the *average time spent* in the system.

First, let's calculate the average arrival rate. The average arrival rate can be calculated by taking the reciprocal of the *mean user think time*. In this case, the mean user think time is given as 5 seconds, so the average arrival rate is 1/5 = 0.2 arrivals per second.

Next, let's calculate the average time spent in the system. The average time spent in the system can be calculated by summing the *expected service time* at device i and the *expected total time* in the system per job. In this case, the expected service time at device i is given as 0.01 seconds and the expected total time in the system per job is given as 50 seconds. So, the average time spent in the system is 0.01 + 50 = 50.01 seconds.

Now, we can calculate the average number of jobs in the queue portion of the CPU by multiplying the average arrival rate by the average time spent in the system. In this case, the average number of jobs in the queue portion of the CPU is 0.2 * 50.01 = 10.002 jobs.

یک سیستم تعاملی که دارای ۱۵ ترمینال ($N = 15$) و میانگین زمان فکر ۵۰ ثانیه ($\mathbb{E}[Z] = 50$) است. بهره‌وری دیسک در این سیستم $\rho_{\text{disk}} = 0.2$ و میانگین تعداد ملاقات‌های دیسک به ازای هر خروجی $\mathbb{E}[V_{\text{disk}}] = 80$ است. اگر میانگین زمان سرویس در هر بار مراجعه به دیسک برابر $\mathbb{E}[S_{\text{disk}}] = 0.025$ ثانیه باشد، امید زمان پاسخ $\mathbb{E}[R]$ را حساب کنید.

Utilization Law: $\rho_i = X_i \cdot \mathbb{E}[S]$	Forced Flow Law: $X_i = \mathbb{E}[V_i] \cdot X$	Little's Law: $N = X \cdot \overline{T}^{\text{Time Avg.}}$.