

Perf Eval of Comp Systems

7.4. Poisson(λ) distribution

$X \sim \text{Poisson}(\lambda)$,

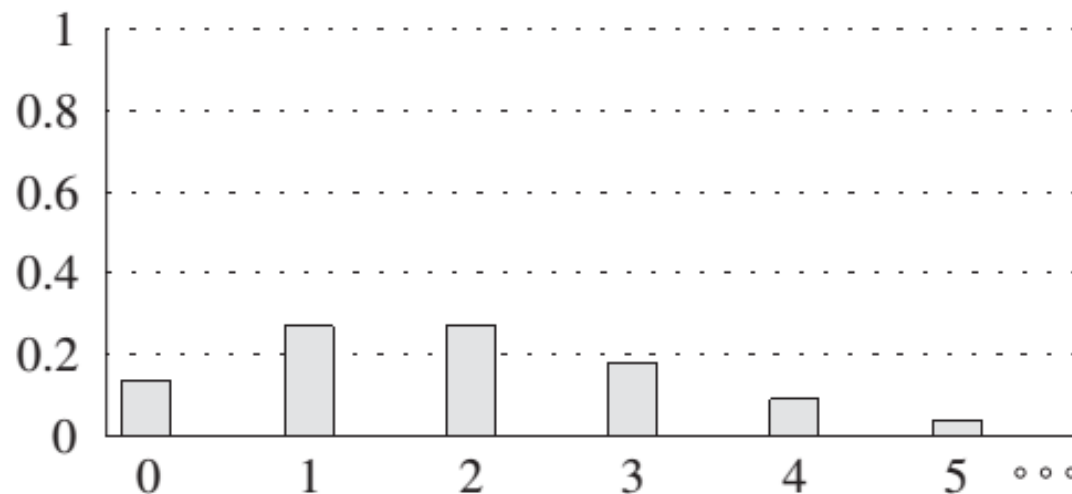
Definition: X is a non-negative integer-valued random variable with the point probabilities

$$p_i = P\{X = i\} = \frac{\lambda^i}{i!} e^{-\lambda} \quad i = 0, 1, \quad (107)$$

Mean and variance are as follows:

$$E[X] = \lambda, \quad V[X] = \lambda. \quad (108)$$

Poisson(λ), $\lambda = 2$



Poisson distribution $X \sim \text{Poisson}(\lambda)$

$$p_i = P\{X = i\} = \frac{\lambda^i}{i!} e^{-\lambda} \quad i = 0, 1, \dots$$

The generating function

$$G(z) = \sum_{i=0}^{\infty} p_i z^i = e^{-\lambda} \sum_{i=0}^{\infty} \frac{(z\lambda)^i}{i!} = e^{-\lambda} e^{z\lambda}$$

$$G(z) = e^{(z-1)\lambda}$$

As we saw before, this generating function is obtained as a limiting form of the generating function of a $\text{Bin}(n, p)$ random variable, when the average number of successes is kept fixed, $np = \lambda$, and n tends to infinity.

Correspondingly, $X \sim \text{Poisson}(\lambda t)$ represents the number of occurrences of events (e.g. arrivals) in an interval of length t from a Poisson process with intensity λ :

- the probability of an event ('success') in a small interval dt is λdt
- the probability of two simultaneous events is $O(\lambda dt)$
- the number of events in disjoint intervals are independent

Poisson distribution (continued)

Poisson distribution is obeyed by e.g.

- The number of arriving calls in a given interval
- The number of calls in progress in a large (non-blocking) trunk group

Expectation and variance

$$E[X] = G'(1) = \left. \frac{d}{dz} e^{(z-1)\lambda} \right|_{z=1} = \lambda \Rightarrow V[x] = E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$E[X^2] = G''(1) + G'(1) = \lambda^2 + \lambda$$

$$E[x] = \lambda \qquad V[x] = \lambda$$

Properties of Poisson distribution

1. The sum of Poisson random variables is Poisson distributed.

$$X = X_1 + X_2, \quad \text{where } X_1 \sim \text{Poisson}(\lambda_1), X_2 \sim \text{Poisson}(\lambda_2)$$

$$\Rightarrow X \sim \text{Poisson}(\lambda_1 + \lambda_2)$$

Proof:

$$G_{X_1}(z) = e^{(z-1)\lambda_1}, G_{X_2}(z) = e^{(z-1)\lambda_2}$$

$$G_X(z) = G_{X_1}(z)G_{X_2}(z) = e^{(z-1)\lambda_1} e^{(z-1)\lambda_2} = e^{(z-1)(\lambda_1 + \lambda_2)}$$

2. If the number, N , of elements in a set obeys Poisson distribution, $N \sim \text{Poisson}(\lambda)$, and one makes a random selection with probability p (each element is independently selected with this probability), then the size of the selected set $K \sim \text{Poisson}(p\lambda)$.

Proof: K obeys the compound distribution

$$K = X_1 + \dots + X_N, \text{ where } N \sim \text{Poisson}(\lambda) \text{ and } X_i \sim \text{Bernoulli}(p)$$

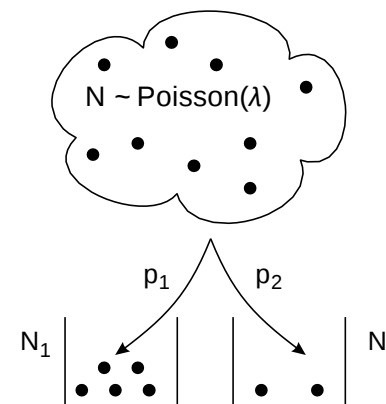
$$G_X(z) = (1 - p) + pz, \quad G_N(z) = e^{(z-1)\lambda}$$

$$G_K(z) = G_N(G_X(z)) = e^{(G_X(z)-1)\lambda} = e^{[(1-p)+pz-1]\lambda} = e^{(z-1)p\lambda}$$

Properties of Poisson distribution (continued)

3. If the elements of a set with size $N \sim \text{Poisson}(\lambda)$ are randomly assigned to one of two groups 1 and 2 with probabilities p_1 and $p_2 = 1 - p_1$, then the sizes of the sets 1 and 2, N_1 and N_2 , are independent and distributed as

$$N_1 \sim \text{Poisson}(p_1\lambda), \quad N_2 \sim \text{Poisson}(p_2\lambda)$$



Proof: By the law of total probability,

$$\begin{aligned}
 P\{N_1 = n_1, N_2 = n_2\} &= \sum_{n=0}^{\infty} \underbrace{P\{N_1 = n_1, N_2 = n_2 | N = n\}}_{\text{multinomial distribution}} \times \underbrace{P\{N = n\}}_{\text{Poisson distribution}} \quad \underbrace{1}_{\text{}} \\
 &= \frac{n!}{n_1! n_2!} p_1^{n_1} p_2^{n_2} \frac{\lambda^n}{n!} e^{-\lambda} \bigg|_{n=n_1+n_2} = \frac{p_1^{n_1} p_2^{n_2}}{n_1! n_2!} \lambda^{n_1+n_2} e^{-\lambda(p_1+p_2)} \\
 &= \frac{(p_1\lambda)^{n_1}}{n_1!} e^{-p_1\lambda} \times \frac{(p_2\lambda)^{n_2}}{n_2!} e^{-p_2\lambda} = P\{N_1 = n_1\} \cdot P\{N_2 = n_2\}
 \end{aligned}$$

The joint probability is of product form $\Rightarrow N_1$ and N_2 independent. The factors in the product are point probabilities of $\text{Poisson}(p_1\lambda)$ and $\text{Poisson}(p_2\lambda)$ distributions.

Note, the result can be generalized for any number of sets.

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7.5 Generating function of the complementary distribution

Let X be a random variable that assumes integer k with probability p_k and let q_k be the distribution for its tails:

$$q_k = P\{X > k\} = p_{k+1} + p_{k+2} + \dots$$

We denote the PGF of $\{p_k\}$ by $P(z)$ and the generating function of $\{q_k\}$ by $Q(z)$. Then it is not difficult to find the following simple relation:

$$Q(z) = \frac{1 - P(z)}{1 - z}$$

Distributions and Their Relationships

